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The properties of functional inclusions and Hyers–Ulam stability

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Abstract. We prove that a set-valued function satisfying some functional inclusions admits, in appropriate conditions, a unique selection satisfying the corresponding functional equation. As a consequence we obtain the result on the Hyers–Ulam stability of that functional equation.

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The first result on the stability of functional equations was given in 1941 by Hyers [7] who proved the following theorem:

Let X be a linear normed space, Y a Banach space and $\epsilon > 0$. Then for every function $f\colon X \to Y$ satisfying the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon, \quad x, y \in X \tag{1}$$

there exists a unique additive function $g\colon X \to Y$ such that

$$\|f(x) - g(x)\| \le \epsilon, \quad x \in X. \tag{2}$$

This was a first answer given to a question proposed by S.M. Ulam in a talk at a conference at the Wisconsin University in 1940 and it represents the starting point of the Hyers–Ulam stability theory of functional equations (see [8,9]). The subject was later strongly developed by many authors, see for example: [1,4,11,12,16]. An interesting connection between the stability of the Cauchy equation and subadditive set-valued functions was established by Smajdor [17] and Gajda and Ger [6]. They observed that if f satisfies (1), then the set-valued function $F\colon X \to n(Y)$ ($n(Y)$ denotes the family of all nonempty subsets of Y) given by

$$F(x) = f(x) + \overline{B}(0, \epsilon), \quad x \in X,$$

where $\overline{B}(0, \epsilon)$ is the closed ball of center 0 and radius ϵ, is subadditive (i.e., $F(x+y) \subset F(x) + F(y), x, y \in X$) and the function g from relation (2) is

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an additive selection of F (i.e., $g(x+y) = g(x) + g(y)$ and $g(x) \in F(x)$ for $x, y \in X$).

The question that appeared as a consequence was: under what conditions a subadditive set-valued map admits an additive selection. An answer to this equation can be found in [6]. Next, the previous result was extended by Nikodem and Popa to set-valued functions satisfying general linear inclusions:

$$F(\alpha x + \beta y + c) \subset \gamma F(x) + \delta F(y) + C,$$
$$\alpha F(x) + \beta F(y) \subset F(\gamma x + \delta y + c) + C,$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, X is a real vector space, Y is a real Banach space, $F\colon X \to n(Y)$, $c \in X$, $C \in 2^Y$ (see [10,13–15]).

It is interesting that we can prove the stability of functional equations corresponding to the functional inclusions considered. This is a motivation to study at first some inclusions (cf. [3]).

Through this paper we assume that K is a nonempty set, Y is a normed space over a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and h is the Hausdorff distance derived from the norm in Y. We will denote by $bd(Y)$ the family of all nonempty and bounded subsets of Y and $bcl(Y)$ stands for the family of all closed sets of $bd(Y)$. The number $\delta(A) = \sup\{\|x-y\| : \ x, y \in A\}$ is said to be the diameter of $A \subset Y$. It easy to see that $\delta(A+B) \le \delta(A) + \delta(B)$ for $A, B \in 2^Y$, where $A + B = \{a + b : \ a \in A,\ b \in B\}$. For $F\colon K \to n(Y)$ we denote by $\operatorname{cl} F$ the multifunction defined as $(\operatorname{cl} F)(x) = \operatorname{cl} F(x), x \in K$. A function $f\colon K \to Y$ such that $f(x) \in F(x)$ for all $x \in K$ is said to be a selection of the multifunction F. We write $a^0(x) = x$ for $x \in K$ and $a^{n+1} = a^n \circ a$ for $n \in \mathbb{N}_0, a\colon K \to K$.

Theorem 1. *Let $F\colon K \to n(Y), \Psi\colon Y \to Y, a\colon K \to K, \lambda \in (0,1)$,*

$$d(\Psi(x), \Psi(y)) \le \lambda d(x,y) \quad \textit{for } x, y \in Y \tag{3}$$

and

$$\sup\{\delta(F(x)) : \ x \in K\} < \infty.$$

(1) *If Y is complete and*

$$\Psi(F(a(x))) \subset F(x), \quad x \in K, \tag{4}$$

then there exists a unique selection f of the multifunction $\operatorname{cl} F$ such that $\Psi \circ f \circ a = f$.

(2) *If*

$$F(x) \subset \Psi(F(a(x))), \quad x \in K, \tag{5}$$

then F is a single-valued function and $\Psi \circ F \circ a = F$.

Proof. (1) Let $x \in K$. Replacing x by $a^n(x)$ in (4) we get

$$\Psi(F(a^{n+1}(x))) \subset F(a^n(x))$$

for all $n \in \mathbb{N}_0$. Hence

$$\Psi^{n+1}(F(a^{n+1}(x))) \subset \Psi^n(F(a^n(x))), \quad n \in \mathbb{N}_0$$

and $(\mathrm{cl}\,\Psi^n(F(a^n(x))))_{n\in\mathbb{N}_0}$ is a decreasing sequence of closed sets in a Banach space. Moreover, in virtue of (3),

$$\delta(\mathrm{cl}\,\Psi^n(F(a^n(x)))) \leq \lambda^n \delta(F(a^n(x))),$$

so $\lim_{n\to\infty} \delta(\mathrm{cl}\,\Psi^n(F(a^n(x)))) = 0$. Therefore

$$\lim_{n\to\infty} \mathrm{cl}\,\Psi^n(F(a^n(x))) = \bigcap_{n\in\mathbb{N}_0} \mathrm{cl}\,\Psi^n(F(a^n(x))) =: f(x)$$

is singleton. Of course $f(x) \in \mathrm{cl}\,F(x)$ and as Ψ is continuous

$$\begin{aligned}\Psi(f(a(x))) &= \Psi\left(\lim_{n\to\infty} \mathrm{cl}\,\Psi^n(F(a^n(a(x)))\right) \subset \lim_{n\to\infty} \mathrm{cl}\,\Psi^{n+1}(F(a^{n+1}(x))) \\ &= f(x),\end{aligned}$$

so $\Psi \circ f \circ a = f$.

It remains to show the uniqueness of f. Suppose that f, g are selections of $\mathrm{cl}\,F$ and $\Psi \circ f \circ a = f, \Psi \circ g \circ a = g$. By induction we obtain that $\Psi^n \circ f \circ a^n = f$ and $\Psi^n \circ g \circ a^n = g$ for $n \in \mathbb{N}_0$. Hence, for $x \in K$,

$$\begin{aligned}d(f(x), g(x)) &= d\left(\Psi^n \circ f \circ a^n(x), \Psi^n \circ g \circ a^n(x)\right) \\ &\leq \lambda^n d(f(a^n(x)), g(a^n(x))) \leq \lambda^n \delta(F(a^n(x))).\end{aligned}$$

Since $\lim_{n\to\infty} \lambda^n \delta(F(a^n(x))) = 0$, we have $f = g$.

(2) By (5) we obtain

$$F(x) \subset \Psi^n(F(a^n(x))) \subset \Psi^{n+1}(F(a^{n+1}(x))), \quad n \in \mathbb{N}_0,\ x \in K.$$

So $(\Psi^n(F(a^n(x))))_{n\in\mathbb{N}_0}$ is an increasing sequence of sets in a normed space with the diameter

$$\delta(\Psi^n(F(a^n(x)))) \leq \lambda^n \delta(F(a^n(x))),$$

which converges to 0 as $n \to \infty$. Therefore $\Psi^n \circ F \circ a^n(x)$ is single-valued for all $n \in \mathbb{N}_0, x \in K$ and $\Psi \circ F \circ a = F$. □

Example. Let $K = Y = \mathbb{R}, f \colon \mathbb{R} \to \mathbb{R}$ be a linear function, $\Psi(x) = \frac{1}{2}x, a(x) = 2x$ and $F(x) = [f(x), f(x) + c]$, where $c \in \mathbb{R}_+$. Then

$$\Psi(F(a(x))) = \left[f(x), f(x) + \frac{c}{2}\right] \subset F(x)$$

and

$$\Psi^n(F(a^n(x))) = \left[f(x), f(x) + \frac{c}{2^n}\right]$$

converges to $f(x)$.

Theorem 2. *Assume that Y is complete, $F, G\colon K \to bd(Y), 0 \in G(x)$ for all $x \in K, \Psi\colon Y \to Y, a\colon K \to K, \lambda \in (0,1)$,*

$$\begin{aligned} &d(\Psi(x), \Psi(y)) \le \lambda d(x,y) \quad \text{for } x, y \in Y,\\ &M := \sup\{\delta(F(x)+G(x)) :\ x \in K\} < \infty \end{aligned}$$

and

$$\Psi(F(a(x))) \subset F(x) + G(x), \quad x \in K. \tag{6}$$

Then there exists a unique function f such that $\Psi \circ f \circ a = f$ and

$$d(f(x), F(x)) \le \frac{1}{1-\lambda} M, \quad x \in K.$$

Proof. Let $x \in K$. Replacing x by $a^n(x)$ in (6) we obtain

$$\Psi(F(a^{n+1}(x))) \subset F(a^n(x)) + G(a^n(x))$$

and, as $0 \in G(x)$, we have

$$F(a^n(x)) \subset F(a^n(x)) + G(a^n(x))$$

for $n \in \mathbb{N}_0$. Thus

$$\begin{aligned} &h(\Psi^{n+1}(F(a^{n+1}(x))), \Psi^n(F(a^n(x))))\\ &\quad \le \lambda^n h(\Psi(F(a^{n+1}(x))), F(a^n(x))) \le \lambda^n \delta(F(a^n(x)) + G(a^n(x))) \end{aligned}$$

for $n \in \mathbb{N}_0$. Hence, for $k \in \mathbb{N}, n \in \mathbb{N}_0$, we obtain

$$\begin{aligned} &h(\Psi^{n+k}(F(a^{n+k}(x))), \Psi^n(F(a^n(x))))\\ &\quad \le \sum_{i=0}^{k-1} \lambda^{n+i} \delta(F(a^{n+i}(x)) + G(a^{n+i}(x)))\\ &\quad = \sum_{j=n}^{n+k-1} \lambda^j \delta(F(a^j(x)) + G(a^j(x))) \le \sum_{j=n}^{n+k-1} \lambda^j M. \end{aligned} \tag{7}$$

Since $\lim_{n\to\infty} \sum_{j=n}^{n+k-1} \lambda^j M = 0$, the sequence $(\Psi^n(F(a^n(x))))_{n\in\mathbb{N}_0}$ is a Cauchy sequence. As $(bcl(Y), h)$ is a complete metric space, there exists the limit $\lim_{n\to\infty} \operatorname{cl} \Psi^n(F(a^n(x)))$. Moreover, the diameter

$$\delta(\operatorname{cl} \Psi^n(F(a^n(x)))) \le \lambda^n \delta(F(a^n(x)))$$

is convergent to 0 as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \operatorname{cl} \Psi^n(F(a^n(x))) =: f(x)$$

is singleton and as Ψ is continuous

$$\begin{aligned} \Psi(f(a(x))) &= \Psi\left(\lim_{n\to\infty} \operatorname{cl} \Psi^n(F(a^n(a(x))))\right) \subset \lim_{n\to\infty} \operatorname{cl} \Psi^{n+1}(F(a^{n+1}(x)))\\ &= f(x). \end{aligned}$$

In this way we have shown that $\Psi \circ f \circ a = f$. By (7), with $n = 0$, we have

$$h(\operatorname{cl}\Psi^k(F(a^k(x))), \operatorname{cl} F(x)) \leq \sum_{j=0}^{k-1} \lambda^j M.$$

Consequently

$$d(f(x), F(x)) \leq \sum_{j=0}^{\infty} \lambda^j M = \frac{1}{1-\lambda} M$$

and

$$f(x) \in F(x) + \frac{1}{1-\lambda} MS,$$

where S is a closed unit ball.

It remains to prove the uniqueness of f. Let f, g be such that $\Psi \circ f \circ a = f, \Psi \circ g \circ a = g$ and $f(x), g(x) \in F(x) + \frac{1}{1-\lambda} MS$ for $x \in K$. By induction we get $\Psi^n \circ f \circ a^n = f$ and $\Psi^n \circ g \circ a^n = g$ for $n \in \mathbb{N}_0$. Hence

$$\begin{aligned} d(f(x), g(x)) &= d(\Psi^n \circ f \circ a^n(x), \Psi^n \circ g \circ a^n(x)) \\ &\leq \lambda^n d(f(a^n(x)), g(a^n(x))) \\ &\leq \lambda^n \delta\left(F(a^n(x)) + \frac{1}{1-\lambda} MS\right), \quad n \in \mathbb{N}. \end{aligned}$$

Since $\lim_{n\to\infty} \lambda^n \delta(F(a^n(x)) + \frac{1}{1-\lambda} MS) = 0$, we have $f = g$. □

In a similar way we obtain

Theorem 3. *Assume that Y is complete, $F, G\colon K \to bd(Y), 0 \in G(x)$ for all $x \in K, \Psi\colon Y \to Y, a\colon K \to K, \lambda \in (0, 1)$,*

$$d(\Psi(x), \Psi(y)) \leq \lambda d(x, y) \quad \textit{for } x, y \in Y,$$
$$M := \sup\{\delta(\Psi(F(a(x))) + G(x)) : \ x \in K\} < \infty$$

and

$$F(x) \subset \Psi(F(a(x))) + G(x), \quad x \in K.$$

Then there exists a unique function f such that $\Psi \circ f \circ a = f$ and

$$d(f(x), F(x)) \leq \frac{1}{1-\lambda} M, \quad x \in K.$$

Example. Let $K = Y = \mathbb{R}, f\colon \mathbb{R} \to \mathbb{R}$ be a linear function, $\Psi(x) = \frac{1}{2}x, a(x) = 2x$ and $F(x) = [f(x) + \sin x, f(x) + c + \sin x]$, where $c \in \mathbb{R}_+$. Then

$$\Psi(F(a(x))) = \left[f(x) + \frac{1}{2}\sin(2x), f(x) + \frac{c}{2} + \frac{1}{2}\sin(2x)\right]$$
$$\subset F(x) + \left[-\sin x - \frac{1}{2}, \frac{3}{2}\right],$$
$$\sup\left\{\delta\left(F(x) + \left[-\sin x - \frac{1}{2}, \frac{3}{2}\right]\right):\ x \in \mathbb{R}\right\} \le c + 3$$

and

$$\Psi^n(F(a^n(x))) = \left[f(x) + \frac{1}{2^n}\sin(2^n x), f(x) + \frac{c}{2^n} + \frac{1}{2^n}\sin(2^n x)\right]$$

converges to $f(x)$.

Corollary 4. *Let Y be complete, $F\colon K \to bd(Y), \Psi\colon Y \to Y, a\colon K \to K, N \in \mathbb{R}, \lambda \in (0, 1)$,*

$$d(\Psi(x), \Psi(y)) \le \lambda d(x, y) \quad \text{for } x, y \in Y,$$
$$M := \sup\{\delta(F(x)):\ x \in K\} < \infty$$

and

$$h(\Psi(F(a(x))), F(x)) \le N, \quad x \in K. \tag{8}$$

Then there exists a unique function f such that $\Psi \circ f \circ a = f$ and

$$d(f(x), F(x)) \le \frac{1}{1-\lambda}(M + 2N), \quad x \in K.$$

Proof. By (8) we have

$$\Psi(F(a(x))) \subset F(x) + NS, \quad F(x) \subset \Psi(F(a(x))) + NS, \quad x \in K,$$

where S is a closed unit ball. Moreover,

$$\sup\{\delta(F(x) + NS):\ x \in K\} \le M + 2N,$$

whence by Theorem 2 or 3 we get the assertion. □

We end the paper with a result on the stability of the equation $\Psi \circ f \circ a = f$.

Theorem 5. *Assume that Y is complete, $f\colon K \to Y, \Psi\colon Y \to Y, a\colon K \to K, k\colon K \to [0, \infty)$ is bounded, $\lambda \in (0, 1)$,*

$$d(\Psi(x), \Psi(y)) \le \lambda d(x, y) \quad \text{for } x, y \in Y$$

and

$$\|\Psi(f(a(x))) - f(x)\| \le k(x) \quad \text{for } x \in K.$$

Then there exists a unique function g such that $\Psi \circ g \circ a = g$ and

$$\|f(x) - g(x)\| \leq \frac{2}{1-\lambda} M, \quad x \in K,$$

where $M = \sup\{|k(x)| : \ x \in K\}$.

Proof. Let $F(x) := \{f(x)\}$. Then

$$\Psi(F(a(x))) \subset F(x) + k(x)S$$

and

$$\sup\{\delta(F(x) + k(x)S) : \ x \in K\} \leq 2M.$$

By Theorem 2 there exists a unique function g such that $\Psi \circ g \circ a = g$ and $\|f(x) - g(x)\| \leq \frac{2}{1-\lambda} M$ for $x \in K$. The assertion follows from Theorem 3 or Corollary 4. $\square$

As it was observed by Forti [5] and explicitly proved in [2], from stability results concerning the equation $\Psi \circ f \circ a = f$ we can easily derive the stability of functional equations in several variables, for example: the Cauchy equation $f(x+y) = f(x) + f(y)$, the Jensen equation $f(\frac{x+y}{2}) = \frac{f(x)+f(y)}{2}$ or the quadratic equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$.

The results that we have been obtained in this paper correspond to the result in [3] and complement them.

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